Chapter 1

Fourier Series

1.1 Introduction.

We will consider complex valued periodic functions on $\mathbf{R}$ with period 2π. We can view them as functions defined on the circumference $\mathbf{T}$ of the unit circle in the complex plane or equivalently as function f defined on $[-\pi, \pi]$ with $f(-\pi) = f(\pi)$. In that case, smoothness requires matching the derivatives as well at $\pm\pi$. The **Fourier Coefficcients** of a periodic function $f \in L_1[-\pi, \pi]$ are defined by

$$a_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx = \frac{1}{2\pi}\int_{\mathbf{T}} f(x)e^{-inx}dx \tag{1.1}$$

If a function f has the representation as a **Fourier Series**

$$f(x) \simeq \sum_{-\infty<n<\infty} a_n e^{inx} \tag{1.2}$$

with $\sum_{-\infty<n<\infty} |a_n| < \infty$, since

$$\frac{1}{2\pi}\int_{\mathbf{T}} e^{imx}e^{-inx}dx = \delta_{n,m}$$

we see that the coefficients a_n can be recovered from f by formula (1.1). If we assume that $f \in L_1[\mathbf{T}]$ then clearly a_n is well defined and

$$|a_n| \le \frac{1}{2\pi}\int_{\mathbf{T}} |f(x)|dx$$

It is not clear that the series on right hand side of equation (1.2) converges and even if it does, it is not clear that the sum of the series is actually equal to the the function $f(x)$. It is relatively easy to find conditions on $f(\cdot)$ so that the series (1.2) is convergent. If $f(x)$ is assumed to be k times continuously differentiable on $\mathbf{T}$, integrating by parts k times one gets, for $n \neq 0$,

$$|a_n| = \frac{1}{|n|^k}\Big|\frac{1}{2\pi}\int_{\mathbf{T}} f^{(k)}(x)e^{-inx}dx\Big| \leq \frac{1}{|n|^k}\sup_x |f^{\{k\}}(x)| \tag{1.3}$$

From the estimate (1.3) it is easily seen that the series is convergent if f is twice continuously differentiable.

An important but elementary fact is the Riemann-Lebesgue theorem.

Theorem 1. *For every $f \in L_1[\mathbf{T}]$,*

$$\lim_{n\to\pm\infty} |a_n| = 0 \tag{1.4}$$

Proof. Let $f \in L_1[\mathbf{T}]$ and $\epsilon > 0$ be given. Since smooth functions are dense in L_1, given $\epsilon > 0$, we can approximate f by a function g_ϵ such that

$$\frac{1}{2\pi}\int_{\mathbf{T}} |f(x) - g_\epsilon(x)|dx \leq \epsilon$$

and g_ϵ is continuously differentiable. Then

$$\begin{aligned}
|a_n| &\leq |\frac{1}{2\pi}\int_{\mathbf{T}} g_\epsilon e^{-inx}dx| + \frac{1}{2\pi}\int_{\mathbf{T}} |f(x) - g_\epsilon(x)|dx \\
&\leq \frac{1}{|n|}\frac{1}{2\pi}\int_{\mathbf{T}} |g'_\epsilon(x)|dx + \frac{1}{2\pi}\int_{\mathbf{T}} |f(x) - g_\epsilon(x)|dx \\
&\leq \frac{1}{|n|}\sup_x |g'_\epsilon(x)| + \epsilon
\end{aligned}$$

and $\limsup_{n\to\pm\infty} |a_n| \leq \epsilon$. Since $\epsilon > 0$ is arbitrary $\limsup_{n\to\pm\infty} |a_n| = 0$. □

1.2 Convergence of Fourier Series.

Let us define the partial sums

$$(s_N f)(x) = s_N(f, x) = \sum_{|n|\leq N} a_n e^{inx} \tag{1.5}$$

and the Fejér sum

$$(S_N f)(x) = S_N(f,x) = \frac{1}{N+1}\sum_{0\le n\le N} s_n(f,x) \tag{1.6}$$

We can calculate

$$\begin{aligned}(s_n f)(x) =& \frac{1}{2\pi}\sum_{|j|\le n} e^{ijx}\int_{\mathbf{T}} e^{-ijy} f(y)dy \\ &= \frac{1}{2\pi}\int_{\mathbf{T}} f(y)[\sum_{|j|\le n} e^{ij(x-y)}]dy \\ &= \frac{1}{2\pi}\int_{\mathbf{T}} f(y)\frac{e^{-in(x-y)}(e^{i(2n+1)(x-y)}-1)}{e^{i(x-y)}-1}dy \\ &= \int_{\mathbf{T}} f(y)k_n(x-y)dy \qquad (1.7) \\ &= (f * k_n)(x) \qquad (1.8)\end{aligned}$$

where

$$k_n(z) = \frac{1}{2\pi}\frac{e^{-inz}(e^{i(2n+1)z}-1)}{e^{iz}-1} = \frac{1}{2\pi}\frac{\sin(n+\frac{1}{2})z}{\sin\frac{z}{2}} \tag{1.9}$$

and the convolution $f * g$ of two functions f, g in $L_1[\mathbf{T}]$ is defined as

$$(f * g)(x) = \int_{\mathbf{T}} f(y)g(x-y)dy = \int_{\mathbf{T}} f(x-y)g(y)dy \tag{1.10}$$

A similar calculation reveals

$$(S_N f)(x) = \int_{\mathbf{T}} f(y)K_N(x-y)dy = (f * K_N)(x) \tag{1.11}$$

where

$$\begin{aligned}K_N(z) =& \frac{1}{2\pi}\frac{1}{(N+1)}\frac{1}{\sin\frac{z}{2}}\sum_{0\le n\le N}[\sin(n+\frac{1}{2})z] \\ &= \frac{1}{2\pi}\frac{1}{(N+1)}\left[\frac{\sin\frac{(N+1)z}{2}}{\sin\frac{z}{2}}\right]^2 \qquad (1.12)\end{aligned}$$

Since

$$k_n(z) = \sum_{|j|\le n} a_j e^{ijx}$$

$$K_N(z) = \frac{1}{N+1}\sum_0^N k_n(z)$$

$$= \sum_{|n|\le N}(1-\frac{|n|}{N+1})a_n e^{i\,n\,x}$$

Notice that for every N,

$$\int_{\mathbf{T}} k_N(z)dz = \int_{\mathbf{T}} K_N(z)dz = 1 \tag{1.13}$$

The following observations are now easy to make.

1. Nonnegativity.
$$K_N(z) \ge 0$$
2. For any $\delta > 0$,
$$\lim_{N\to\infty}\sup_{|z|\ge\delta} K_N(z) = 0$$
3. Therefore
$$\lim_{N\to\infty}\int_{|z|\ge\delta} K_N(z)dz = 0$$

It is now an easy exercise to prove

Theorem 2. *For any f that is bounded and continuous on* $\mathbf{T}$

$$\lim_{N\to\infty}\sup_{x\in\mathbf{T}} |S_N(f,x) - f(x)| = 0$$

Proof. Let $\delta > 0$ be given. Then

$$|S_N(f,x) - f(x)| = |\int [f(x-z) - f(x)]K_N(z)dz|$$

$$\le \int_{|z|\le\delta} |f(x-z) - f(x)|K_N(z)dz + \int_{|z|\ge\delta} |f(x-z) - f(x)|K_N(z)dz$$

$$\le \sup_x \sup_{|z|\le\delta} |f(x-z) - f(x)| + 2\sup_x |f(x)| \int_{|z|\ge\delta} K_N(z)dz$$

If we let $N \to \infty$ and then $\delta \to 0$

$$\limsup_{N\to\infty} \sup_x |S_N(f,x) - f(x)| \le \sup_x \sup_{|z|\le\delta} |f(x-z) - f(x)| \to 0$$

as $\delta \to 0$. $\square$

Theorem 3. *For $1 \le p < \infty$ and $f \in L_p[-\pi,\pi]$*

$$\|S_N(f,\cdot)\|_p \le \|f\|_p$$

and therefore

$$\lim_{N\to\infty} \|S_N f - f\|_p = 0$$

Proof. By Hölder's inequality for any x

$$|S_N(f,x)|^p \le \int_{\mathbf{T}} |f(z)|^p K_N(x-z)dz$$

and integrating with respect to x we obtain the first part of the theorem. For any $f \in L_p$ and $\epsilon > 0$ we can find g such that it is continuous and $\|f-g\|_p \le \epsilon$.

$$\|S_N f - f\|_p \le \|S_N f - S_N g\|_p + \|S_N g - g\|_p + \|g - f\|_p \le \|S_N g - g\|_p + 2\epsilon$$

$\|S_N g - g\|_p \le \sup_x |S_N(g,x) - g(x)| \to 0$ as $N \to \infty$ and $\epsilon > 0$ is arbitrary. $\square$

The behavior of $s_N(f,x)$ is more complicated. It is easy enough to observe that for $f \in C^2(\mathbf{T})$,

$$\lim_{N\to\infty} \sup_x |s_N(f,x) - f(x)| = 0$$

The series converges and so $s_N(f,\cdot)$ has a uniform limit g. The Cesàro average $S_N(f,\cdot)$ has the same limit, which has just been shown to be f. Therefore $f = g$. The following theorem is fairly easy.

Theorem 4. *If $f \in L_1$ satisfies at some x, $|f(y) - f(x)| \le c|y-x|^\alpha$ for some $\alpha > 0$ and $c < \infty$, then at that x,*

$$\lim_{N\to\infty} s_N(f,x) = f(x)$$

Proof. We can assume with out loss of generality that $x = 0$ and let $f(0) = a$. We need to show that

$$\lim_{N\to\infty} \frac{1}{2\pi} \int_{\mathbf{T}} f(y) \frac{\sin(N+\frac{1}{2})y}{\sin\frac{y}{2}} dy = a \tag{1.14}$$

Because $\frac{f(y)-a}{\sin\frac{y}{2}}$ is integrable, (1.14) is a consequence of the Riemann-Lebesgue Theorem, i.e. Theorem 1. $\square$

Let us now assume that f is a function of bounded variation on $\mathbf{T}$ which has left and right limits a_l and a_r at 0. It is easy to check that

$$\left| \frac{1}{\sin(\frac{y}{2})} - \frac{1}{\frac{y}{2}} \right| \le C|y|$$

and it follows from Riemann-Lebesgue theorem that

$$\lim_{\lambda\to\infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) \sin\lambda y \left[\frac{1}{\sin(\frac{y}{2})} - \frac{1}{(\frac{y}{2})} \right] dy = 0$$

By change of variables one can reduce the calculation of

$$\lim_{N\to\infty} \frac{1}{2\pi} \int_{\mathbf{T}} f(y) \frac{\sin(N+\frac{1}{2})y}{\sin\frac{y}{2}} dy$$

to calculating

$$\lim_{\lambda\to\infty} \frac{1}{\pi} \int_{-\lambda\pi}^{\lambda\pi} f\big(\frac{y}{\lambda}\big) \frac{\sin y}{y} dy$$

If we denote by

$$G(y) = \int_y^{\infty} \frac{\sin x}{x} dx$$

then $G(\infty) = 0$ and $G(0) = \frac{\pi}{2}$.

$$\begin{aligned}
a_r(\lambda) &= \frac{1}{\pi} \int_0^{\lambda\pi} f\big(\frac{y}{\lambda}\big) \frac{\sin y}{y} dy = -\frac{1}{\pi} \int_0^{\lambda\pi} f\big(\frac{y}{\lambda}\big) dG(y) \\
&= \frac{1}{2} a_r + \frac{1}{\pi} \int_0^{\lambda\pi} G(y) df\big(\frac{y}{\lambda}\big) = \frac{1}{2} a_r + \frac{1}{\pi} \int_0^{\pi} G(\lambda y) df(y) \\
&\to \frac{1}{2} a_r
\end{aligned}$$

by the bounded convergence theorem. This establishes the following

Theorem 5. *If f is of bounded variation on $\mathbf{T}$, for every $x \in \mathbf{T}$,*

$$\lim_{N\to\infty} s_N(f,x) = \frac{f(x+0)+f(x-0)}{2}$$

On the other hand the behavior of $s_N(f,x)$ for f in $C(\mathbf{T})$, the space of continuous functions on $\mathbf{T}$ or in $L_p[-\pi,\pi]$ for $1 \le p < \infty$ is more complex. For example one can ask. Does $s_N(f,x) \to f(x)$ in L_p ? How about for almost all x? Let us define the linear operator

$$(T_\lambda f)(x) = \int_{-\pi}^{\pi} f(x+y)\frac{\sin \lambda y}{\sin \frac{y}{2}} dy \tag{1.15}$$

on smooth functions f. It is more convenient to think of f as a periodic function of period 2π defined on $\mathbf{R}$. If $s_N(f,x)$ were to converge uniformly to f for every bounded continuous function it would follow by the uniform boundedness principle that

$$\sup_x |(T_\lambda f)(x)| \le C \sup_x |f(x)|$$

with a constant independent of f as well as λ, at least for $\lambda = N + \frac{1}{2}$ where N is a positive integer. Let us show that this is false. The best possible bound $C = C_\lambda$ is seen to be $C_\lambda = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{|\sin\lambda y|}{|\sin \frac{y}{2}|} dy$ and because $|\frac{1}{\sin\frac{y}{2}} - \frac{2}{y}|$ is integrable on $[-\pi,\pi]$, C_λ differs from

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{|2\sin\lambda y|}{|y|} dy = \frac{1}{\pi}\int_{-\lambda\pi}^{\lambda\pi} \frac{|\sin y|}{|y|} dy$$

by a uniformly bounded amount. The divergence of $\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{|\sin y|}{|y|}dy$ implies that $C_\lambda \to \infty$ as $\lambda \to \infty$. By duality this means that $T_\lambda f$ is not uniformly bounded as an operator from $L_1[-\pi,\pi]$ into itself either. Again because of uniform boundedness principle one cannot expect that $s_N(f,\cdot)$ tends to $f(\cdot)$ in $L_1[-\pi,\pi]$ for every $f \in L_1[-\pi,\pi]$.

1.3 Special case $p = 2$

When $p = 2$ we have a Hilbert Space $L_2(\mathbf{T})$ with the inner product

$$\langle f, g\rangle = \frac{1}{2\pi}\int_{\mathbf{T}} \bar{f}\, g\, dx$$

and

$$\|f\|_2^2 = \frac{1}{2\pi} \int_{\mathbf{T}} |f(x)|^2 dx$$

We have taken the normalized Lebesgue measure $d\mu = \frac{dx}{2\pi}$ so that $\mu(T) = 1$. The functions $\{e_n(\cdot) = e^{inx} : n \in Z\}$ form a complete orthonormal basis and the Fourier series is the expansion

$$f(x) = \sum_{n=-\infty}^{\infty} a_n e^{inx}$$

which converges in $L_2(\mathbf{T})$ with a_n given by

$$a_n = \langle e_n, f \rangle = \frac{1}{2\pi} \int_{\mathbf{T}} e^{-inx} f(x) dx$$

The Plancherel-Perseval identities state

$$\sum_{n=-\infty}^{\infty} |a_n|^2 = \frac{1}{2\pi} \int_{\mathbf{T}} |f(x)|^2 dx$$

and

$$\sum_{n=-\infty}^{\infty} \bar{a}_n b_n = \frac{1}{2\pi} \int_{\mathbf{T}} \overline{f(x)} g(x) dx$$

1.4 Higher dimensions.

If we have periodic functions $f(\mathbf{x}) = f(x_1, \ldots, x_d)$ of d variables with period 2π in every variable then the Fourier transforms are defined on $\mathbf{Z}^d$. If $\mathbf{n} = (n_1, \ldots, n_d)$ then

$$a_{\mathbf{n}} = \frac{1}{(2\pi)^d} \int_{\mathbf{T}^d} e^{i\langle \mathbf{n}, \mathbf{x} \rangle} d\mathbf{X}$$

While most of the one dimensional results carry over to d dimensions, one needs to be careful about the partial sums. Results that depend on the explicit form of the kernels k_N and K_N have to reexamined. While partial sums over sets of the form $\cap_i \{|n_i| \le N\}$ or even $\cap_i \{|n_i| \le N_i\}$ can be handled, it is hard to analyze partial sums over sets of the form $\{\mathbf{n} : \sum_i n_i^2 \le N\}$.

1.5 Singular Integrals.

We start with a useful covering lemma known as Vitali covering lemma.

Lemma 6. *Let $K \subset S$ be a compact subset of* $\mathbf{R}$ *and $\{I_\alpha\}$ be a collection of intervals covering K. Then there is a finite sub-collection $\{I_j\}$ such that*

1. *$\{I_j\}$ are disjoint.*
2. *The intervals $\{3I_j\}$ that have the same midpoints as $\{I_j\}$ but three times the lenghth cover K.*

Proof. We first choose a finite subcover. From the finite subcover we pick the largest interval. In case of a tie pick any of the competing ones. Then, at any stage, of the remaining intervals from our finite subcollection we pick the largest one that is disjoint from the ones already picked. We stop when we cannot pick any more. The collection that we end up with is clearly disjoint and finite. Let $x \in K$. This is covered by one of the intervals I from our finite subcollection covering K. If I was picked there is nothing to prove. If I was not picked it must intersect some I_j already picked. Let us look at the first such interval and call it $\overline{I}$. I is disjoint from all the previously picked ones and I was passed over when we picked $\overline{I}$. Therefore in addition to intersecting $\overline{I}$, I is not larger than $\overline{I}$. Therefore $3\overline{I} \supset I \ni x$. □

The lemma is used in proving maximal inequalities. For instance, for the Hardy-Littlewood maximal function we have

Theorem 7. *Let $f \in L_1(\mathbf{T})$. Define*

$$M_f(x) = \sup_{0<r<\frac{\pi}{2}} \frac{1}{2r} \int_{|y-x|<r} |f(y)|dy \tag{1.16}$$

$$\mu[x : M_f(x) > \ell] \leq \frac{3 \int |f(y)|dy}{\ell} \tag{1.17}$$

Proof. Let us denote by E_ℓ the set

$$E_\ell = \{x : M_f(x) > \ell\}$$

and let $K \subset E_\ell$ be an arbitrary compact set. For each $x \in K$ there is an interval I_x such that

$$\int_{I_x} |f(y)|dy \geq \ell\mu(I_x)$$

Clearly $\{I_x\}$ is a covering of K and by lemma we get a finite disjoint sub collection $\{I_j\}$ such that $\{3I_j\}$ covers K. Adding them up

$$\int |f(y)|dy \geq \ell \sum_j \mu(I_j) = \frac{\ell}{3} \sum_j \mu(3I_j) \geq \frac{\ell}{3}\mu(K)$$

Sine $K \subset E_\ell$ is arbitrary we are done. ☐

There is no problem in replacing $\{x : |M_f(x)| > \ell\}$ by $\{x : |M_f(x)| \geq \ell\}$. Replace ℓ by $\ell - \epsilon$ and let $\epsilon \to 0$.

This theorem can be used to prove the Labesgue diffrentiability theorem.

Theorem 8. *For any $f \in L_1(S)$,*

$$\lim_{h\to 0} \frac{1}{2h} \int_{|x-y|\leq h} |f(y) - f(x)|dy = 0 \qquad \text{a.e.} \quad x \tag{1.18}$$

Proof. It is sufficient to prove that for any $\delta > 0$

$$\mu[x : \limsup_{h\to 0} \frac{1}{2h} \int_{|x-y|\leq h} |f(y) - f(x)|dy \geq \delta] = 0$$

Given $\epsilon > 0$ we can write $f = f_1 + g$ with f_1 continuous and $\|g\|_1 \leq \epsilon$ and

$$\begin{aligned}
\mu[x : \limsup_{h\to 0} \frac{1}{2h} \int_{|x-y|\leq h} |f(y) - f(x)|dy \geq \delta]& \\
&= \mu[x : \limsup_{h\to 0} \frac{1}{2h} \int_{|x-y|\leq h} |g(y) - g(x)|dy \geq \delta] \\
&\leq \mu[x : \sup_{h>0} \frac{1}{2h} \int_{|x-y|\leq h} |g(y) - g(x)|dy \geq \delta] \\
&\leq \frac{3\|g\|_1}{\delta} \leq \frac{3\epsilon}{\delta}
\end{aligned}$$

Since $\epsilon > 0$ is arbitrary we are done. ☐

In other words the maximal inequality is useful to prove almost sure convergence. Typically almost sure convergence will be obvious for a dense set and the maximal inequality will be used to interchange limits in the approximation.

Another summability method, like the Fejer sum that is often considered is the Poisson sum

$$S(\rho, x) = \sum_n a_n \rho^{|n|} e^{inx}$$

and the kernel corresponding to it is the Poisson kernel

$$p(\rho, z) = \frac{1}{2\pi} \sum_n \rho^{|n|} e^{inz} = \frac{1}{2\pi} \frac{1-\rho^2}{(1 - 2\rho\cos z + \rho^2)} \tag{1.19}$$

so that

$$P(\rho, x) = \int f(y) p(\rho, x - y) dy$$

It is left as an exercise to prove that for for $1 \le p < \infty$, every $f \in L_p$ $P(\rho, \cdot) \to f(\cdot)$ in L_p as $\rho \to 1$. We will prove a maximal inequality for the Poisson sum, so that as a consequence we will get the almost sure convergence of $P(\rho, x)$ to f for every f in L_1.

Theorem 9. *For every f in L_1*

$$\mu[x : \sup_{0 \le \rho < 1} P(\rho, x) \ge \ell] \le \frac{C\|f\|_1}{\ell} \tag{1.20}$$

Proof. The proof consists of estimating the Poisson maximal function in terms of the Hardy-Littlewood maximal function $M_f(x)$.

We begin with some simple estimates for the Poisson kernel $p(\rho, z)$.

$$\begin{aligned} p(\rho, z) =& \frac{1}{2\pi} \frac{1-\rho^2}{(1-\rho)^2 + 2\rho(1-\cos z)} \le \frac{1}{2\pi} \frac{1-\rho^2}{(1-\rho)^2} \\ =& \frac{1}{2\pi} \frac{1+\rho}{1-\rho} \le \frac{1}{\pi} \frac{1}{1-\rho} \end{aligned}$$

The problem therefore is only as $\rho \to 1$. Lets us assume that $\rho \ge \frac{1}{2}$.

Lemma 10. *For any symmetric function $\phi(z)$ the integral*

$$
\begin{aligned}
&|\int_{-\pi}^{\pi} f(z)\phi(z)dz| \\
&\quad = |\int_0^{\pi} [f(z)+f(-z)]\phi(z)dz| \\
&\quad = |\int_0^{\pi} \phi(z)[\frac{d}{dz}\int_{-z}^{z} f(y)dy]dz| \\
&\quad \leq |\int_0^{\pi} \phi'(z)[\int_{-z}^{z} f(y)dy]dz| + |\phi(\pi)\int_{-\pi}^{\pi} f(z)dz| \\
&\quad \leq \int_0^{\pi} 2|z\phi'(z)|\frac{1}{2z}[\int_{-z}^{z} |f(y)|dy]dz + \phi(\pi)|\int_{-\pi}^{\pi} |f(z)|dz \\
&\quad \leq 2M_f(0)\int_0^{\pi} |z\phi'(z)|dz + |\phi(\pi)||M_f(0)|
\end{aligned}
$$

For the Poisson kernel

$$
\begin{aligned}
|z\frac{d}{dz}p(\rho,z)| &= \frac{1}{2\pi}\frac{1-\rho^2}{(1-2\rho\cos z+\rho^2)^2}2\rho|z\sin z| \\
&\leq \frac{1}{\pi}\frac{(1-\rho)z^2}{(1-\rho)^4+(1-\cos z)^2} \\
&\leq C\frac{(1-\rho)z^2}{(1-\rho)^4+z^4}
\end{aligned}
$$

and

$$
\begin{aligned}
\int_{-\pi}^{\pi} |z\frac{d}{dz}p(\rho,z)|dz &\leq C\int_{-\pi}^{\pi} \frac{(1-\rho)z^2}{(1-\rho)^4+z^4}dz \\
&= \int_{-\frac{\pi}{1-\rho}}^{\frac{\pi}{1-\rho}} \frac{z^2}{1+z^4}dz \\
&\leq \int_{-\infty}^{\infty} \frac{z^2}{1+z^4}dz \leq C_1
\end{aligned}
$$

uniformly in ρ.

$\square$

1.6 Exercises.

1. Instead of Fejér sum if we use

$$(W_N f)(x) = \sum a_n w(N,n) e^{i\,n\,x}$$

   with $w(N,n) \to 1$ as $N \to \infty$ what simple additional conditions will ensure the convergence of $W_N f$ to f? In $L_1[\mathbf{T}]$ or $L_2[\mathbf{T}]$

2. What about $w(N,n) = e^{-\frac{n}{N}}$?

3. What about $w(N,n) = e^{-\frac{n^2}{N}}$?

4. What about $w(N,n) = 2e^{-\frac{n^2}{N}} - e^{-\frac{2n^2}{N}}$

5. Can you formulate higher dimensional analogs ?

6. Use lemma 10 to complete the proof of Theorem 9.

7. For an function f defined on $\mathbf{T}$, the harmonic extension inside the circle is given by

$$U(f,r,\alpha) = \frac{1}{2\pi} \int \frac{f(\theta)}{(1 - 2r\cos(\theta - \alpha) + r^2)} d\theta$$

   Show that for $f \in L_1[\mathbf{T}]$

$$\lim_{r \to 1} U(f,r,\alpha) = f(\alpha)$$

   for almost all $\alpha \in \mathbf{T}$